THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW10 Solution

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1. (P. 276 Q1c)

   Let $x_n = 2^{-\frac{1}{n}}$. Since $\lim\limits_{n\to\infty} x_n = 1 \neq 0$, by the contrapositive of 3.7.3 of the textbook, the series diverges.

2. (P. 276 Q3c)

   Note that

$$\sum_{n=1}^{\infty} (\ln n)^{-\ln n} = \sum_{m=0}^{\infty} \sum_{n=2^m}^{2^{m+1}-1} (\ln n)^{-\ln n} \leq \sum_{m=0}^{\infty} \sum_{n=2^m}^{2^{m+1}-1} (\ln 2^m)^{-\ln 2^m}$$
$$= \sum_{m=0}^{\infty} 2^m (\ln 2^m)^{-\ln 2^m}$$

   We aim to show that $\sum_{m=0}^{\infty} 2^m (\ln 2^m)^{-\ln 2^m}$ converges: let $x_m = 2^m (\ln 2^m)^{-\ln 2^m}$. Note that

$$x_m = \frac{2^m}{(m \ln 2)^{m \ln 2}} = \left(\frac{2}{(m \ln 2)^{\ln 2}}\right)^m$$

   Since $\lim\limits_{m\to\infty} \frac{2}{(m \ln 2)^{\ln 2}} = 0$, there exists $M \in \mathbb{N}$ such that for all $m \geq M$, $\frac{2}{(m \ln 2)^{\ln 2}} < \frac{1}{2}$. Therefore, for all $m \geq M$, $x_m \leq \left(\frac{1}{2}\right)^m$. Therefore, by Comparison Test (3.7.7 of the textbook), $\sum_{m=0}^{\infty} 2^m (\ln 2^m)^{-\ln 2^m}$ converges, and hence by the first inequality, $\sum_{n=1}^{\infty} (\ln n)^{-\ln n}$ converges.

   Note: The trick in the first inequality can be generalised to a test known as "Cauchy condensation test", which is particularly useful when the series involves logarithm.

3. (P. 276 Q4c)

   Note that $e^{-\ln n} = e^{\ln(n^{-1})} = \frac{1}{n}$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, the series diverges.

4. (P. 280 Q9)

   Let $x_n = e^{-nt}$ and $y_n = a_n$. Then since x_n is decreasing with $\lim\limits_{n\to\infty} x_n = 0$, and by assumption $\sum a_n$ is bounded, by Dirichlet Test (9.3.4 of the textbook), $\sum x_n y_n = \sum a_n e^{-nt}$ converges.